Control of an Inverted Pendulum

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1 Introduction

The single inverted pendulum is a classical problem in the field of non-linear control theory [1]. Essentially, the system as it is discussed here consists of a cart and a pendulum attached to it as illustrated in figure 1. Applying a force to the cart, for instance through a built-in electrical motor, and thus moving it forwards and backwards will cause the pendulum to swing. However, modelling these dynamics yields a highly non-linear problem. It is the aim of the controller to apply the right amount of force which will stabilize the pendulum in the upright position. This can either be performed with the pendulum being held upright or through an upswing manouevre.

One approach in finding a suitable control law is to linearize the system equations e. g. around the upright position. As the linearized model does not represent the real system for larger deviations this is not an appropriate approach for a complete upswing and stabilization control. It is however possible to design an independent second controller which performs the upswing movement and then hands over to the linear controller.

An intuitive way of swinging the pendulum to the upright position is to consider the energy stored in the system and comparing it to the value which corresponds to the maximum height [2]. This energy control approach yields a switching control law, similar to the design of a sliding mode control [3]. Sliding mode control is a very powerful design method and its advantages can be benchmarked using the example of a single inverted pendulum [4].

Possible enhancements to the model discussed in this report could be to consider the cart's electrical motor dynamics as well. The control input would then be the voltage applied to the motor rather than the motor's force. A second variation is the rotational inverted pendulum where the pendulum is mounted on a rotational basis rather than a cart [5].

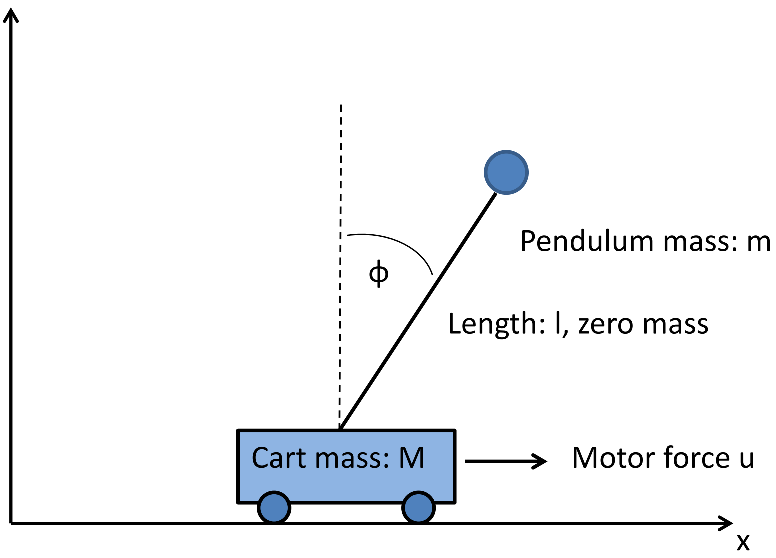

Figure 1: Single inverted pendulum mounted on cart.

2 Dynamics Equations for the Inverted Pendulum

The inverted pendulum considered in this report can be described by two non-linear differential equations. These can be derived using the Euler-Lagrange equation. The non-linear model with M being the mass of the cart, m the mass of the pendulum's bob, l the pendulum's length, θ the angle between the pendulum and its upright position and x the position of the cart is:

$$(M+m)\ddot{x} + b\dot{x} - ml\sin(\theta)\dot{\theta}^2 + ml\cos(\theta)\ddot{\theta} = u \tag{1}$$

$$m\ddot{x}\cos(\theta) + ml\ddot{\theta} = mg\sin(\theta) \tag{2}$$

2.1 Deriving a non-linear state space model

From these equations, a non-linear state-space model can be derived:

$$\frac{d}{dt}\begin{pmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} \dot{\theta} \\ g_1(\theta, \theta, x, \dot{x}, u) \\ \dot{x} \\ g_2(\theta, \theta, x, \dot{x}, u) \end{pmatrix} \tag{3}$$

with

$$g_1(\theta, \theta, x, \dot{x}, u) = \frac{(M+m)\,g\sin(\theta) + b\cos(\theta)\,\dot{x} - ml\sin(\theta)\cos(\theta)\,\dot{\theta}^2 - \cos(\theta)\,u}{l\left(M + m\sin^2(\theta)\right)} \tag{4}$$

and

$$g_2(\theta, \theta, x, \dot{x}, u) = \frac{-mg\sin(\theta)\cos(\theta) - b\dot{x} + ml\sin(\theta)\,\dot{\theta}^2 + u}{M + m\sin^2(\theta)}\ . \tag{5}$$

This is accomplished by first solving (2) for $\ddot{x}$ and $\ddot{\theta}$ respectively:

$$\ddot{x} = \frac{g\sin(\theta) - l\ddot{\theta}}{\cos(\theta)} \tag{6}$$

$$\ddot{\theta} = \frac{g\sin(\theta) - \ddot{x}\cos(\theta)}{l}\ . \tag{7}$$

Inserting these expressions into (1) gives:

$$\begin{aligned}
u &= (M+m)\frac{g\sin(\theta) - l\ddot{\theta}}{\cos(\theta)} + b\dot{x} - ml\sin(\theta)\dot{\theta}^2 + ml\cos(\theta)\ddot{\theta} \\
u &= \ddot{\theta}\left(ml\cos(\theta) - \frac{(M+m)l}{\cos(\theta)}\right) + b\dot{x} - ml\sin(\theta)\dot{\theta}^2 \\
\ddot{\theta} &= \left(-\frac{\cos(\theta)}{Ml + ml\sin^2(\theta)}\right)\left(-b\dot{x} - +ml\sin(\theta)\dot{\theta}^2 + u\right) \\
\ddot{\theta} &= \frac{(M+m)\,g\sin(\theta) + b\cos(\theta)\,\dot{x} - ml\sin(\theta)\cos(\theta)\,\dot{\theta}^2}{l\left(M + m\sin^2(\theta)\right)} - \frac{\cos(\theta)\,u}{l\left(M + m\sin^2(\theta)\right)}
\end{aligned}$$

and

$$\begin{aligned}
u &= (M+m)\ddot{x} + b\dot{x} - ml\sin(\theta)\dot{\theta}^2 + ml\cos(\theta)\frac{g\sin(\theta) - \ddot{x}\cos(\theta)}{l} \\
u &= \ddot{x}\left((M+m) - m\cos^2(\theta)\right) + b\dot{x} - ml\sin(\theta)\dot{\theta}^2 + mg\cos(\theta)\sin(\theta) \\
\ddot{x} &= \left(\frac{1}{M + m\sin^2(\theta)}\right)\left(-b\dot{x} + ml\sin(\theta)\dot{\theta}^2 - mg\cos(\theta)\sin(\theta) + u\right) \\
\ddot{x} &= \frac{ml\sin(\theta)\dot{\theta}^2 - mg\cos(\theta)\sin(\theta) - b\dot{x}}{M + m\sin^2(\theta)} + \frac{u}{M + m\sin^2(\theta)}\ .
\end{aligned}$$

2.2 Deriving a linearized state space model

In order to find a suitable control for this non-linear system, a linearized system model is developed. Firstly, the approximations $\sin(\theta) \approx \theta$ and $\cos(\theta) \approx 1$ for small values of θ are inserted in equations (1) and (2), resulting in:

$$(M+m)\,\ddot{x} + b\dot{x} - ml\theta\dot{\theta}^2 + ml\ddot{\theta} = u \tag{8}$$

$$m\ddot{x} + ml\ddot{\theta} = mg\theta\ . \tag{9}$$

Now, taylor-series approximation is used to find a linear model for the pertubations Δx and $\Delta\theta$ from a given linearization point $\theta_0, \dot{\theta}_0$. Later, this point will be set to $\theta_0 = 0,\ \dot{\theta}_0 = 0$ which also conforms with the above approximations.

$$\frac{d}{dt}\Delta\dot{x} = \frac{1}{(M+m)}\left(-b\Delta\dot{x} + ml\dot{\theta_0}^2\Delta\theta + 2ml\theta_0\dot{\theta}_0\Delta\dot{\theta} - ml\frac{d}{dt}\Delta\dot{\theta} + \Delta u\right) \tag{10}$$

$$\frac{d}{dt}\Delta\dot{\theta} = \frac{1}{l}\left(g\Delta\theta - m\frac{d}{dt}\Delta\dot{x}\right) \tag{11}$$

In order to find a state-space model, further transformations have to be applied to these equations, which yields:

$$\frac{d}{dt}\Delta\dot{x} = -\frac{b}{M}\cdot\Delta\dot{x} + \frac{m\left(l\dot{\theta_0}^2 - g\right)}{M}\cdot\Delta\theta + \frac{2ml\theta_0\dot{\theta}_0}{M}\cdot\Delta\dot{\theta} + \frac{1}{M}\cdot\Delta u \tag{12}$$

$$\frac{d}{dt}\Delta\dot{\theta} = \frac{b}{lM}\cdot\Delta\dot{x} + \left(\frac{g(M+m)}{lM} - \frac{m\dot{\theta_0}^2}{M}\right)\cdot\Delta\theta - \frac{2m\theta_0\dot{\theta}_0}{M}\cdot\Delta\dot{\theta} - \frac{1}{lM}\cdot\Delta u\ . \tag{13}$$

This approximation does not depend on the assumption $\theta_0 = 0,\ \dot{\theta}_0 = 0$ so far but on the less strict condition that $\theta_0 \lll 1$. Of course, all pertubations Δ are assumed to be small in order for the first-order taylor-series approximation to hold. Inserting $\theta_0 = 0,\ \dot{\theta}_0 = 0$ as it would be caused by the pendulum being manually held in the upright position yields a further simplied model:

$$\frac{d}{dt}\Delta\dot{x} = -\frac{b}{M}\cdot\Delta\dot{x} - \frac{mg}{M}\cdot\Delta\theta + \frac{1}{M}\cdot\Delta u \tag{14}$$

$$\frac{d}{dt}\Delta\dot{\theta} = \frac{b}{lM}\cdot\Delta\dot{x} + \frac{g(M+m)}{lM}\cdot\Delta\theta - \frac{1}{lM}\cdot\Delta u\ . \tag{15}$$

Therefore, the linear state space model is:

$$\frac{d}{dt}\begin{pmatrix}\Delta\theta\\ \Delta\dot{\theta}\\ \Delta x\\ \Delta\dot{x}\end{pmatrix} = \begin{pmatrix}0 & 1 & 0 & 0\\ \frac{g(M+m)}{lM} & 0 & 0 & \frac{b}{lM}\\ 0 & 0 & 0 & 1\\ -\frac{mg}{M} & 0 & 0 & -\frac{b}{M}\end{pmatrix}\begin{pmatrix}\Delta\theta\\ \Delta\dot{\theta}\\ \Delta x\\ \Delta\dot{x}\end{pmatrix} + \begin{pmatrix}0\\ -\frac{1}{lM}\\ 0\\ \frac{1}{M}\end{pmatrix}\Delta u\ . \tag{16}$$

Inserting $M = 3kg$, $m = 0.2kg$, $l = 0.31m$ and $b = 0.1N/ms^{-1}$ yields:

$$\frac{d}{dt}\begin{pmatrix}\Delta\theta\\ \Delta\dot{\theta}\\ \Delta x\\ \Delta\dot{x}\end{pmatrix} = \begin{pmatrix}0 & 1 & 0 & 0\\ 33.75 & 0 & 0 & 0.11\\ 0 & 0 & 0 & 1\\ -0.65 & 0 & 0 & -0.03\end{pmatrix}\begin{pmatrix}\Delta\theta\\ \Delta\dot{\theta}\\ \Delta x\\ \Delta\dot{x}\end{pmatrix} + \begin{pmatrix}0\\ -1.08\\ 0\\ 0.33\end{pmatrix}\Delta u\ . \tag{17}$$

Using MATLAB to determine the eigenvalues of this linear state space model yields

$$\lambda = \begin{pmatrix} 5.8084 & 0 & -0.0279 & -5.8105 \end{pmatrix} \tag{18}$$

and due to the eigenvalues in the right half-plane the system is unstable.

2.3 Deriving a discrete time linear state space model

A discrete time linear state space with time steps of $T = 1.25\,ms$ model can be found using MATLABs continuous-to-discrete (c2d) command:

$$\begin{pmatrix} \triangle\theta_{k+1} \\ \Delta\dot{\theta}_{k+1} \\ \Delta x_{k+1} \\ \Delta\dot{x}_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 0.0013 & 0 & 0 \\ 0.0430 & 1 & 0 & 0.0001 \\ 0 & 0 & 1 & 0.0012 \\ -0.0008 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \triangle\theta_{k} \\ \Delta\dot{\theta}_{k} \\ \Delta x_{k} \\ \Delta\dot{x}_{k} \end{pmatrix} + \begin{pmatrix} 0 \\ -1.35\cdot 10^{-3} \\ 0 \\ 0.4125\cdot 10^{-3} \end{pmatrix} \Delta u_k\,. \tag{19}$$

3 Control Strategy for Linearized System

In this section, a discrete time state variable feedback control is designed for the previously outlined linearized system model. The design criteria are a 2% settling time of 2 seconds and 20% overshoot. These parameters will be realised by setting the position of the two dominant poles, whereas the remaining poles will be laid further into the left half-plane in order to minimize their influence.

3.1 Closed-loop pole locations

The continuous time characteristic equation for the dominant poles is given as

$$s^2 + 2\varsigma\omega_n s + \omega_n^2 = 0 \ . \tag{20}$$

The parameter values are determined using the following formulas:

$$P_{overshoot,\%} = 100 \exp\left(\frac{-\pi\varsigma}{\sqrt{1-\varsigma^2}}\right) \tag{21}$$

$$T_{settling,2\%} = \frac{4}{\varsigma\omega_n} \ . \tag{22}$$

This yields $\varsigma = 0.455$ and $\omega_n = 4.396$. In general, the pole locations of equation (20) can be obtained as

$$s_{1,2} = -\varsigma\omega_n \pm j\omega_n\sqrt{1-\varsigma^2} \ . \tag{23}$$

Inserting the values calculated before, this gives

$$s_{1,2} = -2 \pm j3.9142 \ . \tag{24}$$

Using a separation factor of 3, the remaining poles will be located at

$$s_{3,4} = -6 \pm j11.7426 \ . \tag{25}$$

In order to find a control law for the discrete time system given in equation (19), the coresponding poles in z-domain will be calculated first and then Ackerman's formula will be used to obtain a feedback matrix k.

Transformation from s-domain to z-domain is performed using

$$z = \exp(sT) \tag{26}$$

where in this case a sampling time of $T = 1.25\,ms$ is chosen. For a second order system, the poles in z-domain are thus located at

$$z_{1,2} = \exp\left(-\varsigma\omega_n T \pm j\omega_n T\sqrt{1-\varsigma^2}\right) \ . \tag{27}$$

Here, this leads to

$$z_{1,2} = 0.9975 \pm j0.0049 \tag{28}$$

and

$$z_{3,4} = 0.9924 \pm j0.0146 \ . \tag{29}$$

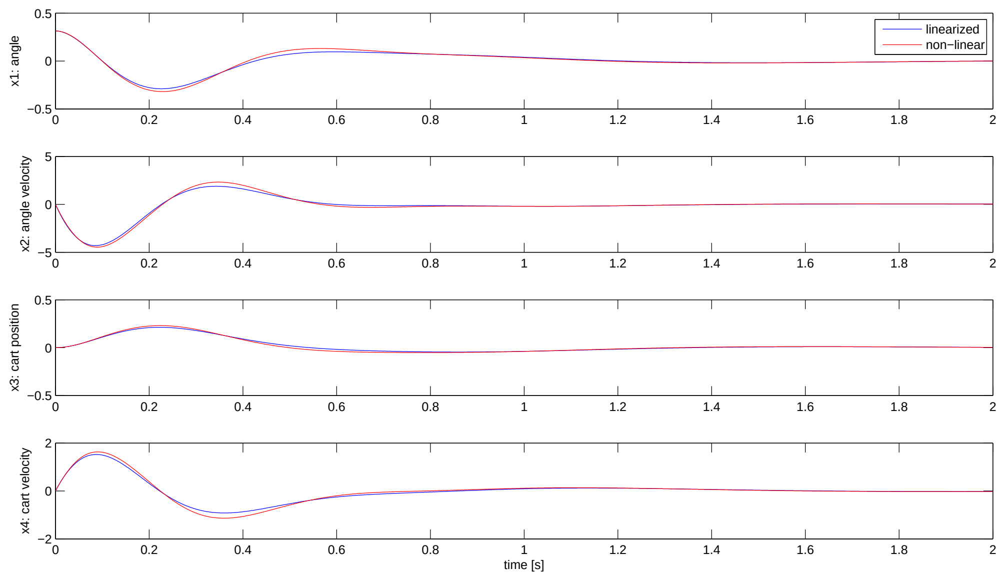

Figure 2: Simulation with intial angle $\theta = 0.1\pi$

3.2 State feedback control law

Ackerman's formula or the MATLAB command `place` can be used to find the following control law for the linearized discrete time system 19:

$$\Delta \underline{x}_{k+1} = A \Delta \underline{x}_k - bK \Delta \underline{x}_k \tag{30}$$

with

$$K = \begin{pmatrix} -352.2366 & -41.5126 & -312.2914 & -85.9176 \end{pmatrix} . \tag{31}$$

At this point it is assumed that all state variables are known. Otherwise, a separate observer or Kalman filter had to be designed in order to gain the state variables from measurement variables (e. g. a linear observer with observer gains derived from the linearized model [6]).

3.3 Simulation of the linear control design and lab implementation

The simulation is performed using MATLAB 7.12. Simulating this control on both the linearized and the original non-linear model yields the results shown in figure 2 and 3. While in the first case an initial angle of $\theta = 0.1\pi = 18°$ was chosen, the second time the pendulum was set to the downward position with $\theta = \pi$. Although the control seems to work for both cases on the linearized model, the true non-linear dynamics show instability when starting from the downward position. Therefore this controller is only suitable if the pendulum is moved to the upright position first, either manually or by a second controller.

Further testing reveals that oscillations occur when an initial angle greater than 0.17π is set. This control only works for a range of initial angles $[-30.6°, 30.6°]$. Concerning the saftey of the implemented system, a condition could be applied to automatically disable all control inputs if the angle exceeds this range of valid angles.

When deriving the linearized system model, the approximations $\sin(\theta) \approx \theta$ and $\cos(\theta) \approx 1$ for small values of θ are used. This causes the linearized form to be accurate only for small angles and will otherwise result in deviations to the real system. Moreover, the model was further simplified by setting $\theta_0 = 0$ which means that the linearized equations will model the dynamics around this point. In total, the linearized model does not sufficiently represent the dynamics for initial angles greater than

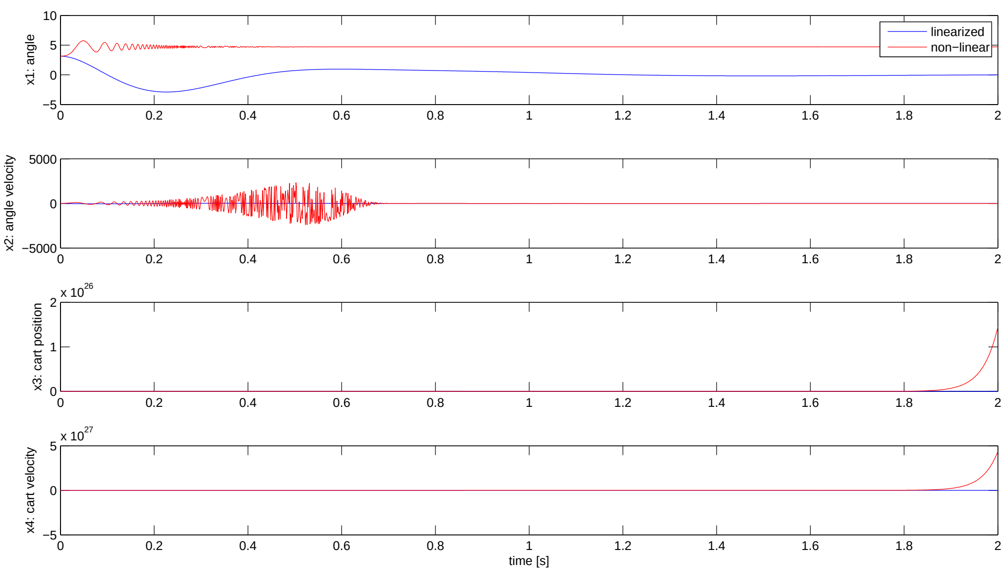

Figure 3: Simulation with intial angle $\theta = \pi$

$\frac{\pi}{3}$ anymore. The control law is designed to move the eigenvalues of one particular system to the left half-plane and therefore fails when applying it to an entirely different system.

Implementing the previously described control law on the real pendulum proves that this design is capable of stabilizing the pendulum in the upright position. Although the simulation results suggest the pendulum and cart staying at a fixed position after a short settling time, this is not true for the real system. Rather the cart moves constantly within a range of 30 cm to maintain stability. A possible explanation for this unforseen behaviour is that the model does not perfectly represent the system. And because the system itself is unstable, any small deviation will increase and therefore the controller has to react.

4 Non-linear Sliding Mode Control

The sliding mode controller is capable of controlling the pendulum over all angles and is therefore suitable for upswing control. In this part, a hybrid approach will be used with a sliding mode controller for upswing and a linear state feedback controller based on the linearized model for angles around the upright position.

4.1 Deriving a Sliding Mode Control law

For the sliding mode controller, the sliding surface

$$x_2 = -ax_1 \tag{32}$$

is used. It follows from the definition $x_2 = \dot{x}_1$ that

$$\dot{x}_1 = -ax_1 \tag{33}$$

which is solved as

$$x_1(t) = x_1(0)\exp(-at) \ . \tag{34}$$

This means that $\theta(t) = x_1(t)$ converges to zero and therefore a stable manifold has been defined. Note that only the pendulum's angle and angle velocity are considered here, a controller which also includes the cart's position and velocity would require a higher order surface. Later, tuning the parameters will be used to adjust the cart's movement to stay within certain boundaries. It is the aim of the sliding mode controller to reach this manifold. The distance from the surface is calculated as

$$S = ax_1 + x_2 \ . \tag{35}$$

From this a Lyapunov function

$$V = \frac{1}{2}S^2 \tag{36}$$

can be found with

$$\dot{V} = S\dot{S} = S(ax_2 + \dot{x}_2) \ . \tag{37}$$

$$= S\left(a\dot{\theta} + \frac{d}{dt}\dot{\theta}\right) \tag{38}$$

From Lyapunov theory follows that the sliding surface will be reached as long as the derivative $\dot{V}$ remains negative definite. In order to accomplish this a control law depending on the sign of S is derived.

The non-linear system equation (4) can be rewritten as

$$\dot{\theta} = f(\underline{x}) + ug(\underline{x}) \tag{39}$$

with

$$f(\underline{x}) = \frac{(M+m)g\sin(\theta) + b\cos(\theta)\dot{x} - ml\sin(\theta)\cos(\theta)\dot{\theta}^2}{l\left(M + m\sin^2(\theta)\right)} \tag{40}$$

and

$$g(\underline{x}) = \frac{-\cos(\theta)}{l\left(M + m\sin^2(\theta)\right)} \ . \tag{41}$$

Inserting these expressions in equation (37) yields

$$\dot{V} = S\left(a\dot{\theta} + f(\underline{x}) + ug(\underline{x})\right) \tag{42}$$

Negative definiteness is therefore ensured by the control law

$$u = \frac{1}{g(\underline{x})}\left(-a\dot{\theta} - f(\underline{x})\right) - k \cdot sign(S) \ . \tag{43}$$

The linear part of the controller takes over at $\theta = 0.1\pi = 18°$ and uses the linearized system model from equations (12) and (13). In contrast to the previous result (16), now initial values θ_0, $\dot{\theta}_0$ other than zero are incorporated. The linearized state space model thus takes the system state at the beginning of the control take-over into account which should result in optimized control behaviour. The dominant eigenvalues are chosen to be identical real values (critically damped case)

$$\lambda_{1,2} = -3 \tag{44}$$

and the remaining poles will be located at

$$\lambda_3 = -8 \ \ \lambda_4 = -9 \ . \tag{45}$$

The controller parameters are set to $k = 125$ and $a = 3.3$.

4.2 Simulation of the non-linear control design

Results of this controller for a complete upswing and stabiliziation are shown in figure 4.

It has to be noted that a switching control law where the sign of the control input depends on the system states' value in comparison to the sliding surface "chattering" can occur. This is because the control law always strives for the system being driven towards the sliding surface. However, as it will never reach the surface exactly but rather always end up on the opposite side the control input will rapidly switch for system states close to the surface. One way of reducing this unwanted behavious is to insert a tolerance band around the exact sliding surface. As long as the system state stays within the tolerance band (regardless on which side in relation to the surface), no control input is given. Here, a deadband of ± 0.1 is implemented and the results are shown in figure (5). Comparing the two phaseplane plots clearly shows that chattering has been eliminated.

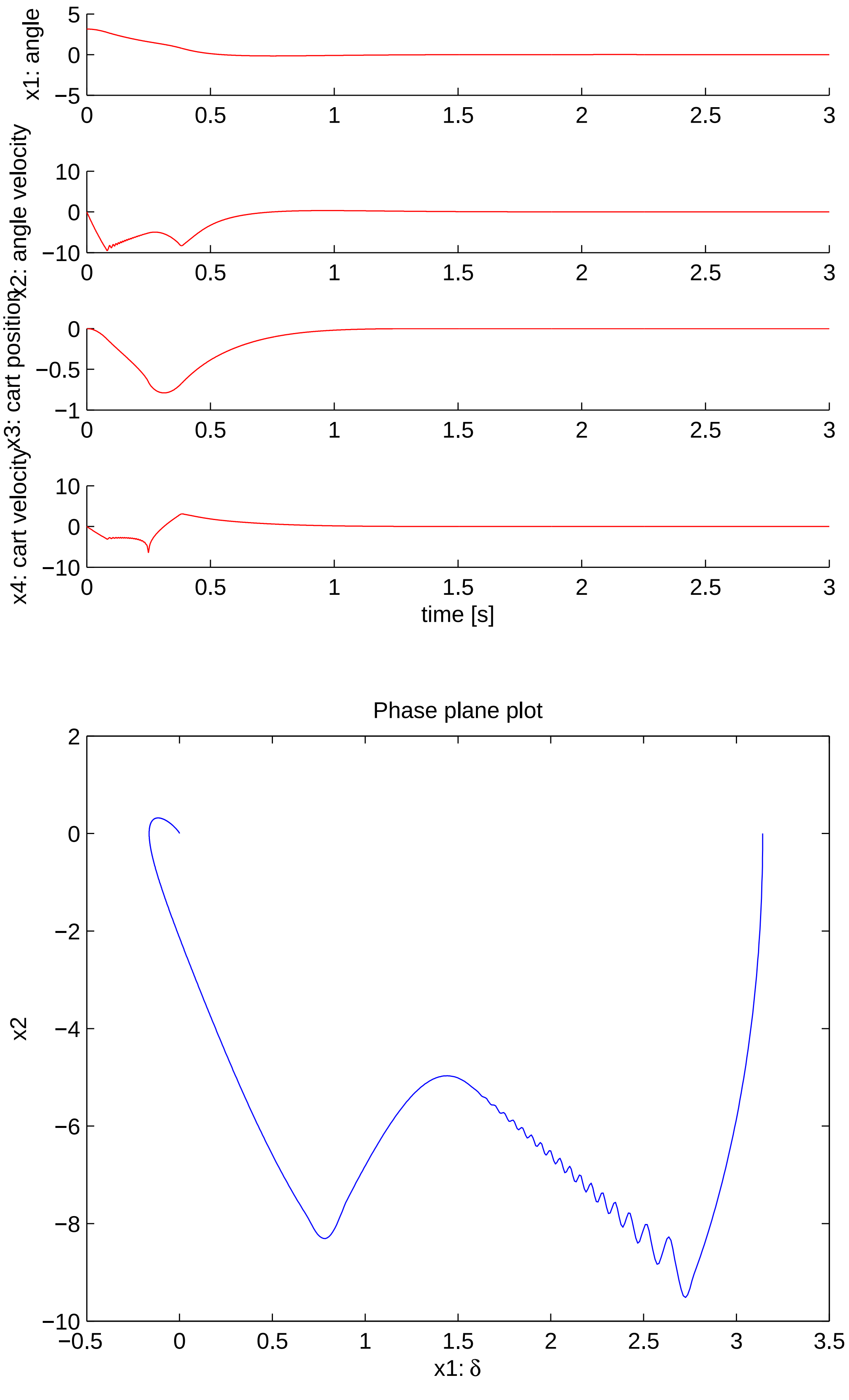

Figure 4: Results of hybrid control using both Sliding Mode and linearized control

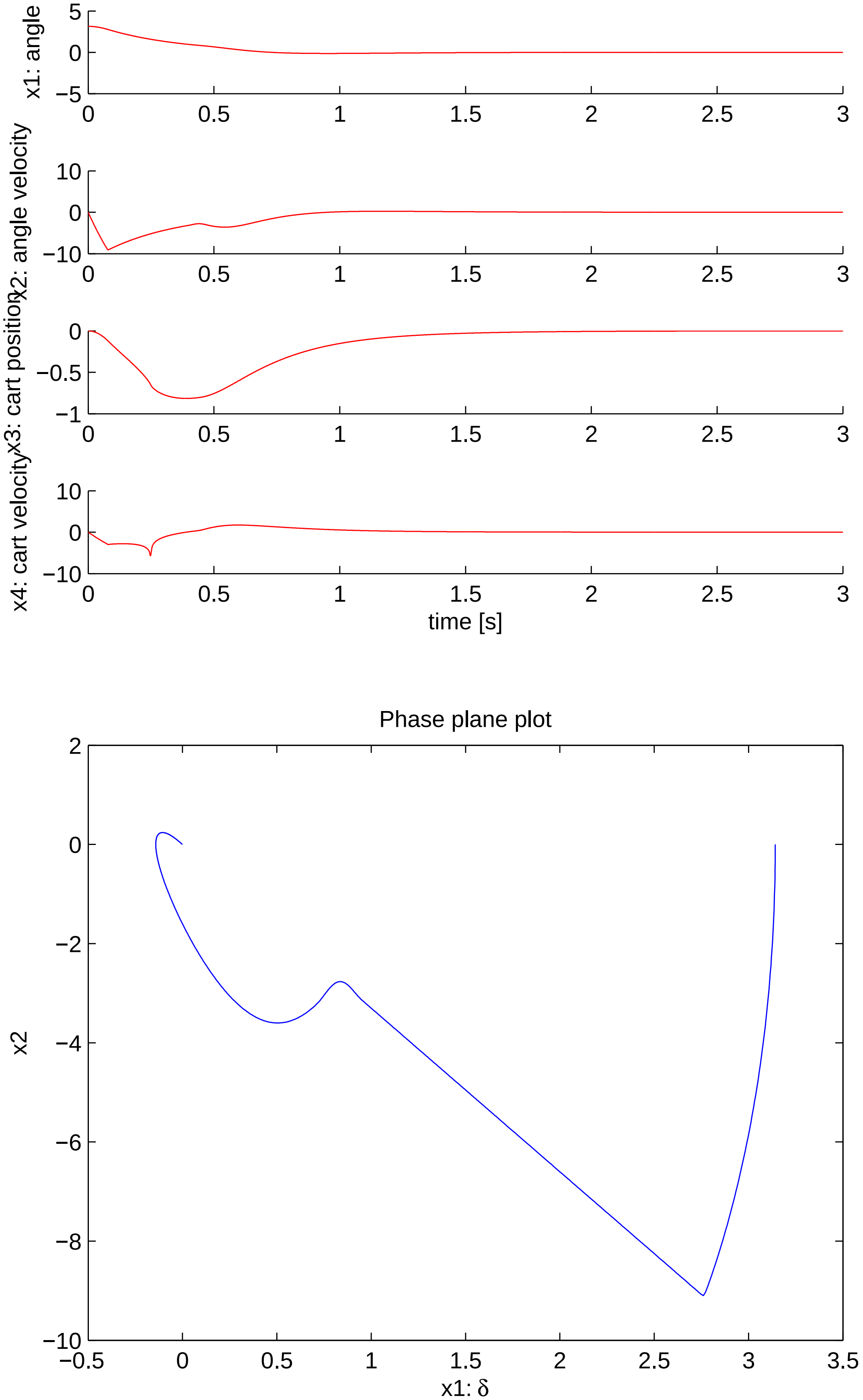

Figure 5: Results of hybrid control with anti-chattering measures.

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